Pseudorandom Generators

CS/ECE 407

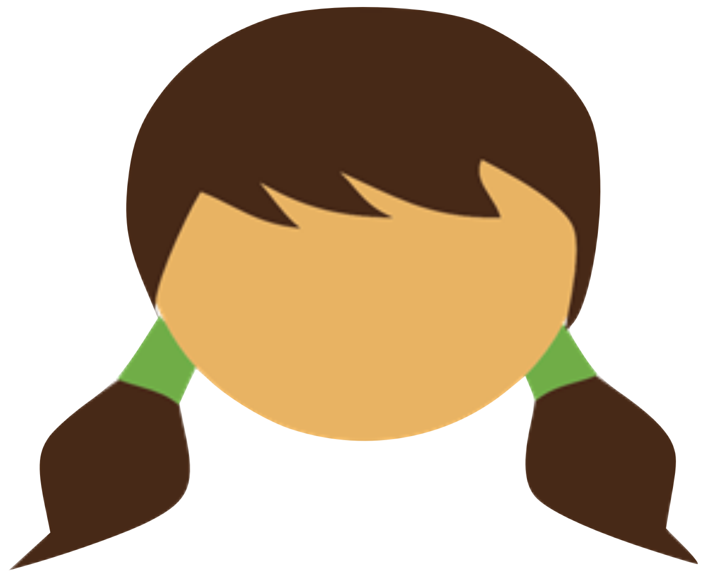
Today's objectives

Describe pseudorandomness/pseudorandom generators

Define negligible functions

Introduce indistinguishability

ct

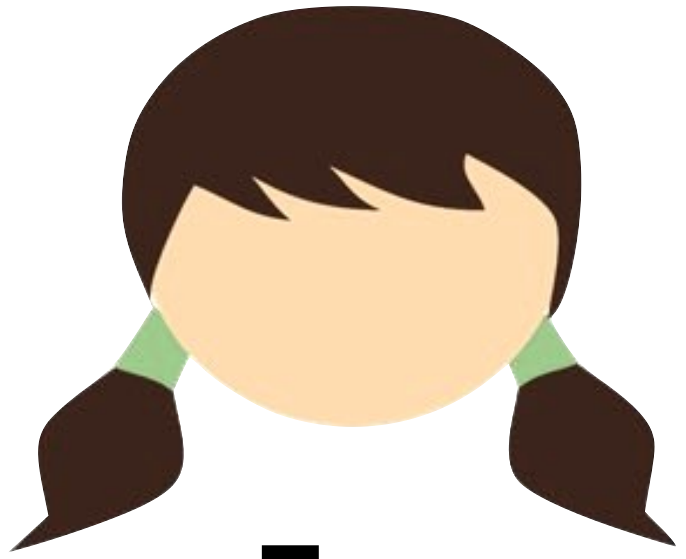


Alice

$m \in \{0,1\}$

$k \leftarrow_{\$} \{0,1\}$

$ct \leftarrow m \oplus k$



Eve

Bob

$k \leftarrow_{\$} \{0,1\}$

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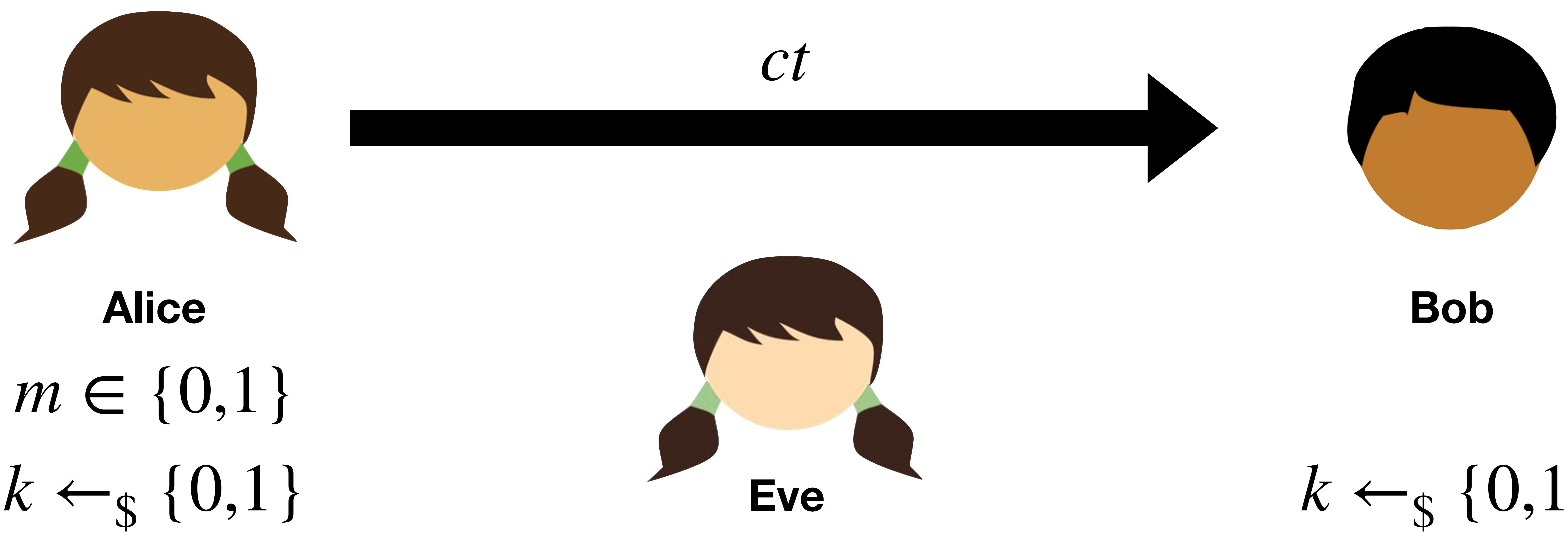

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Question: *what if Alice wants to send more than one bit?*

Perfect Secrecy:

A cipher (Enc, Dec) is **perfectly secret** if for every message $m \in \mathrm{M}$:

$$\left\{ c \;\middle|\; \begin{array}{l} k \leftarrow_{\$} K \\ c = Enc(k, m) \end{array} \right\} \equiv \left\{ c \;\middle|\; c \leftarrow_{\$} C \right\}$$

Theorem [Shannon 1949]: *Any cipher achieving perfect secrecy requires that* $|\mathrm{K}| \geq |\mathrm{M}|$*.*

“If we want to encrypt more stuff, we need more randomness”

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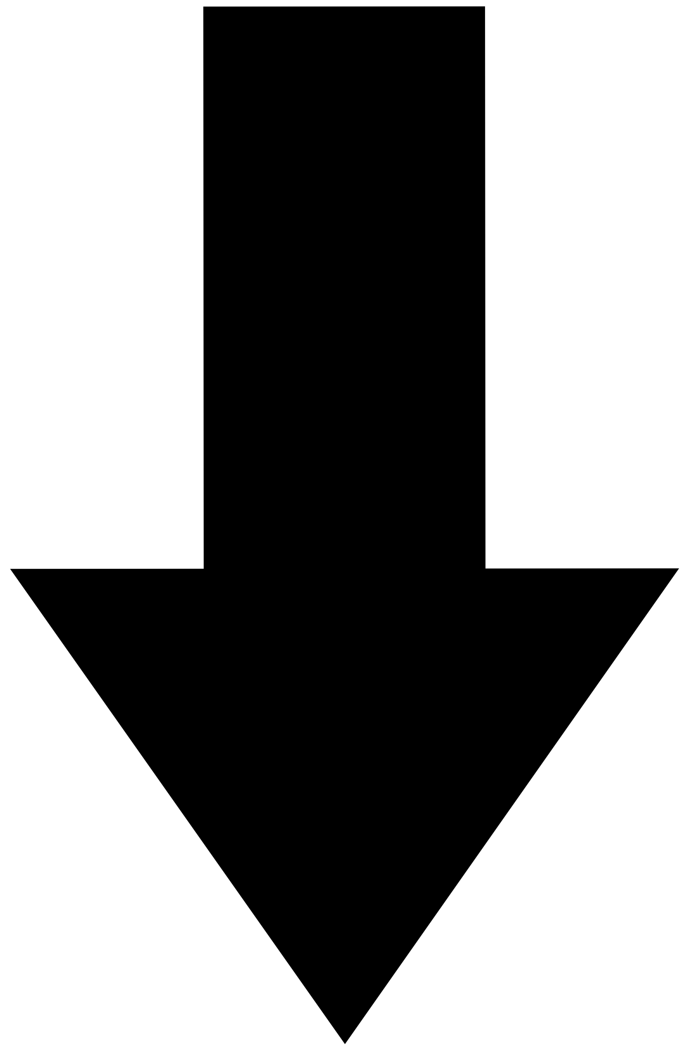

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Q: Can we turn a short random string into a long random string?

“If we want to encrypt more stuff, we need more randomness”

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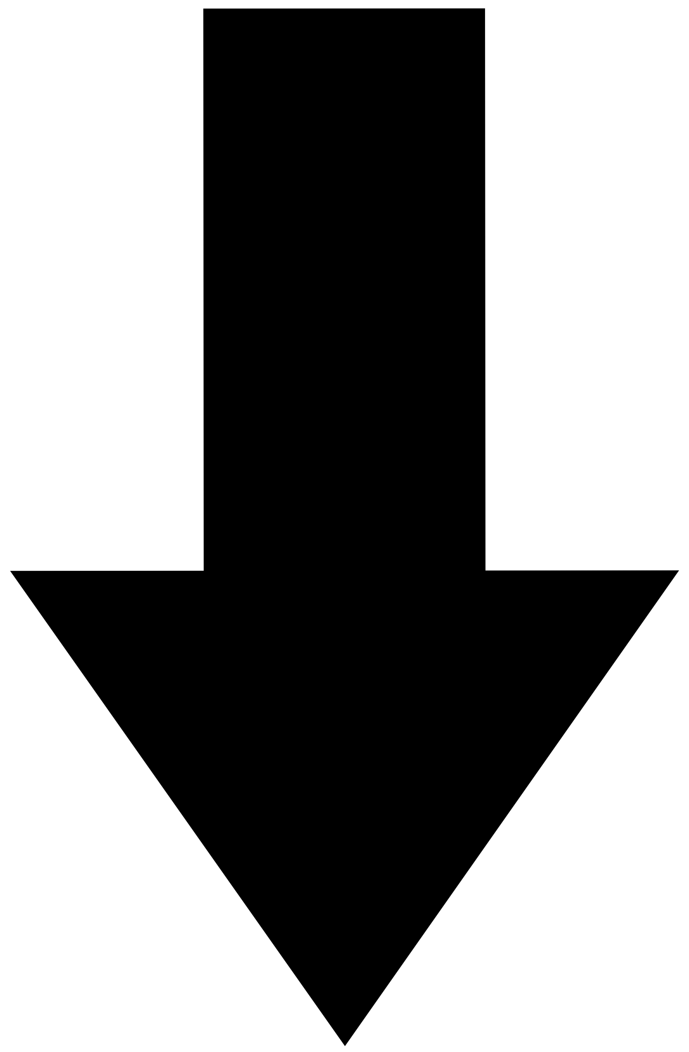

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A: No, this is impossible

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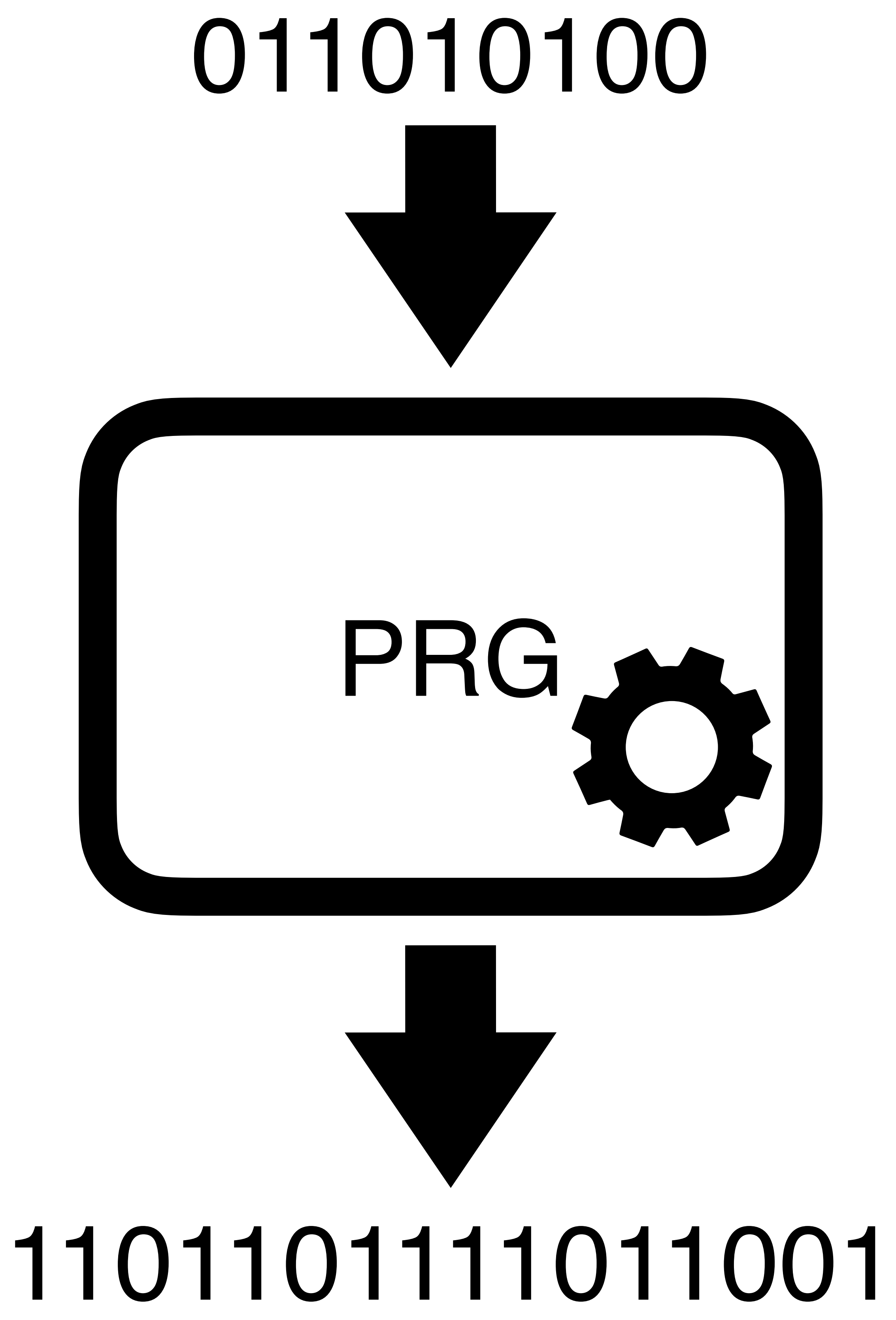

Q: Can we turn a short random string into a long random string?

A: No, this is impossible

Q: Can we turn a short random string into a long string that <u>looks</u> random?

A: Yes[†]! Use a pseudorandom generator!

† Or, at least, we believe this to be possible

Pseudorandom Generator (PRG)

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Informal: *"no program can tell the difference between the output of* G *and truly random strings"*

Hardness as a basis for cryptography

Security?

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Modern Cryptography

State assumptions

Define security

Design system

Prove: if assumption holds, system meets definition

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State assumptions **PRGs exist**

Define security

Design system

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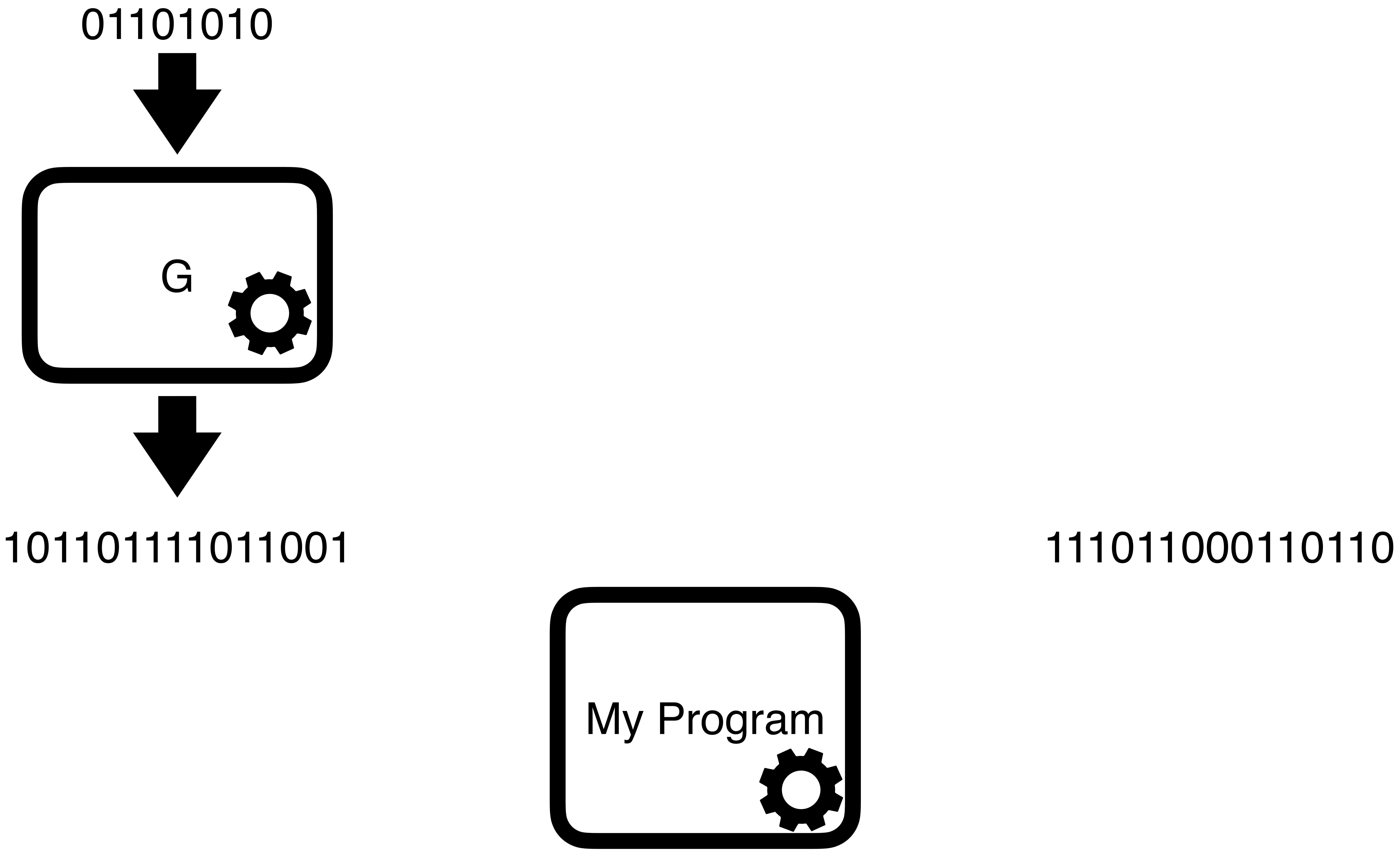
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G
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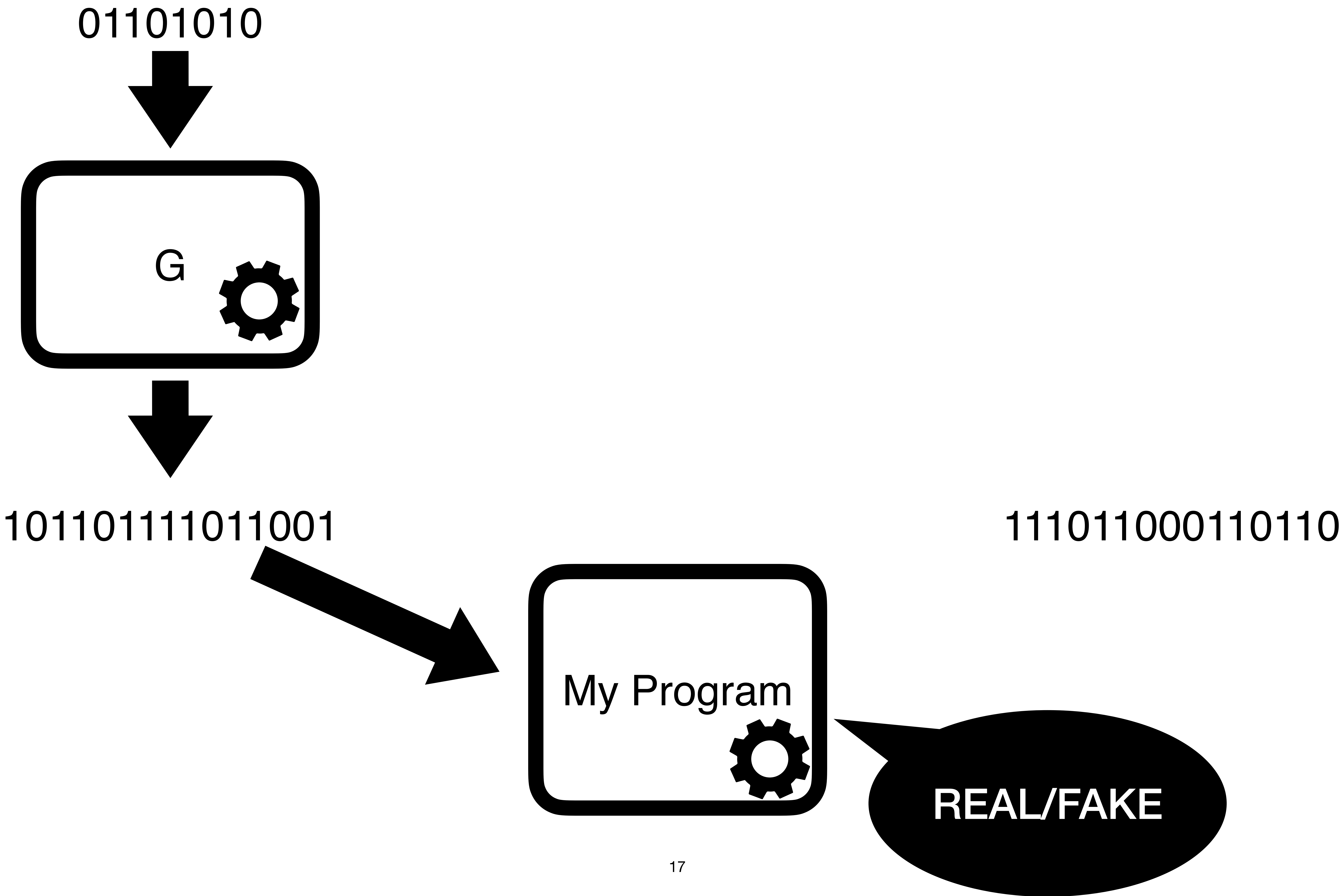
My Program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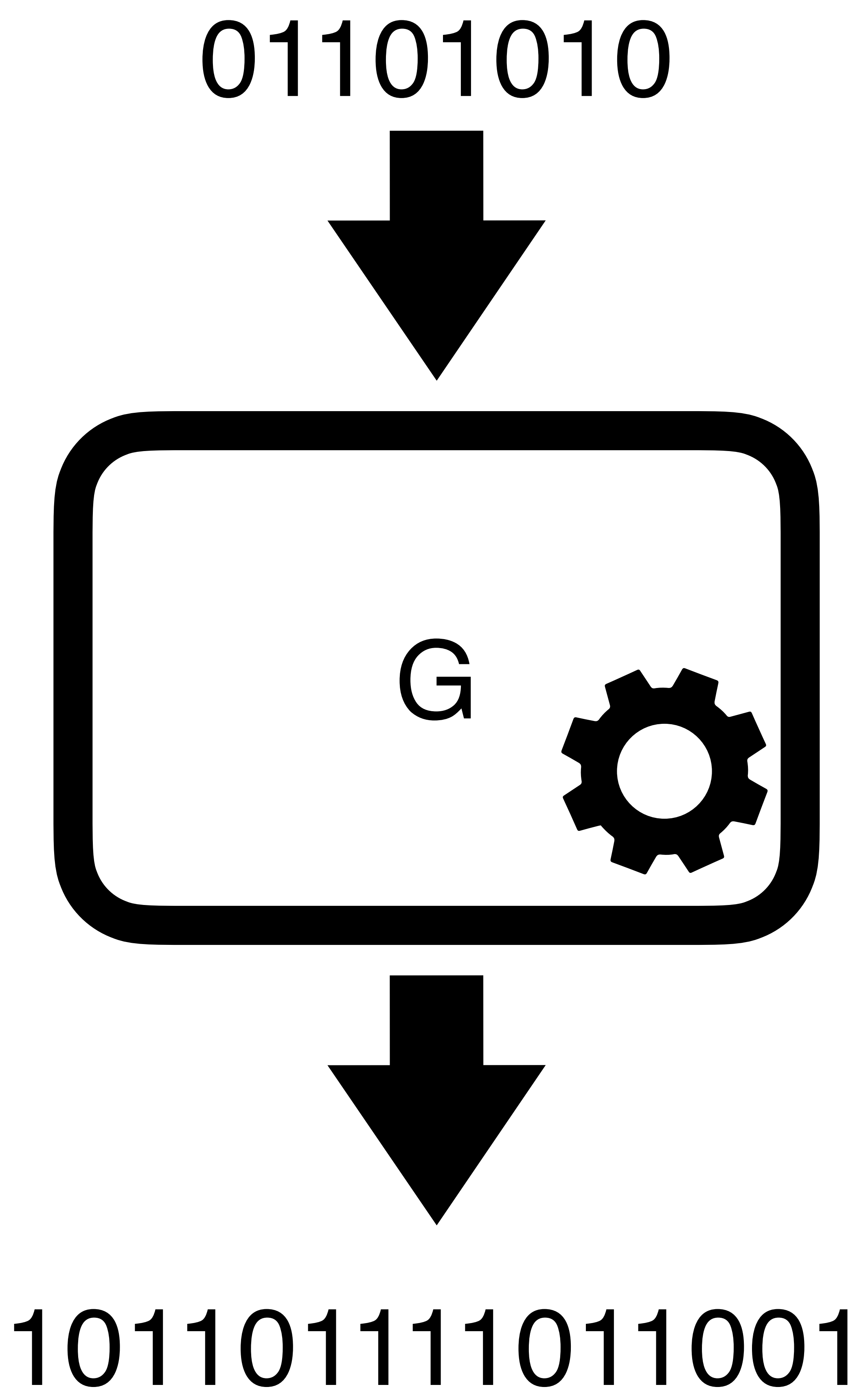
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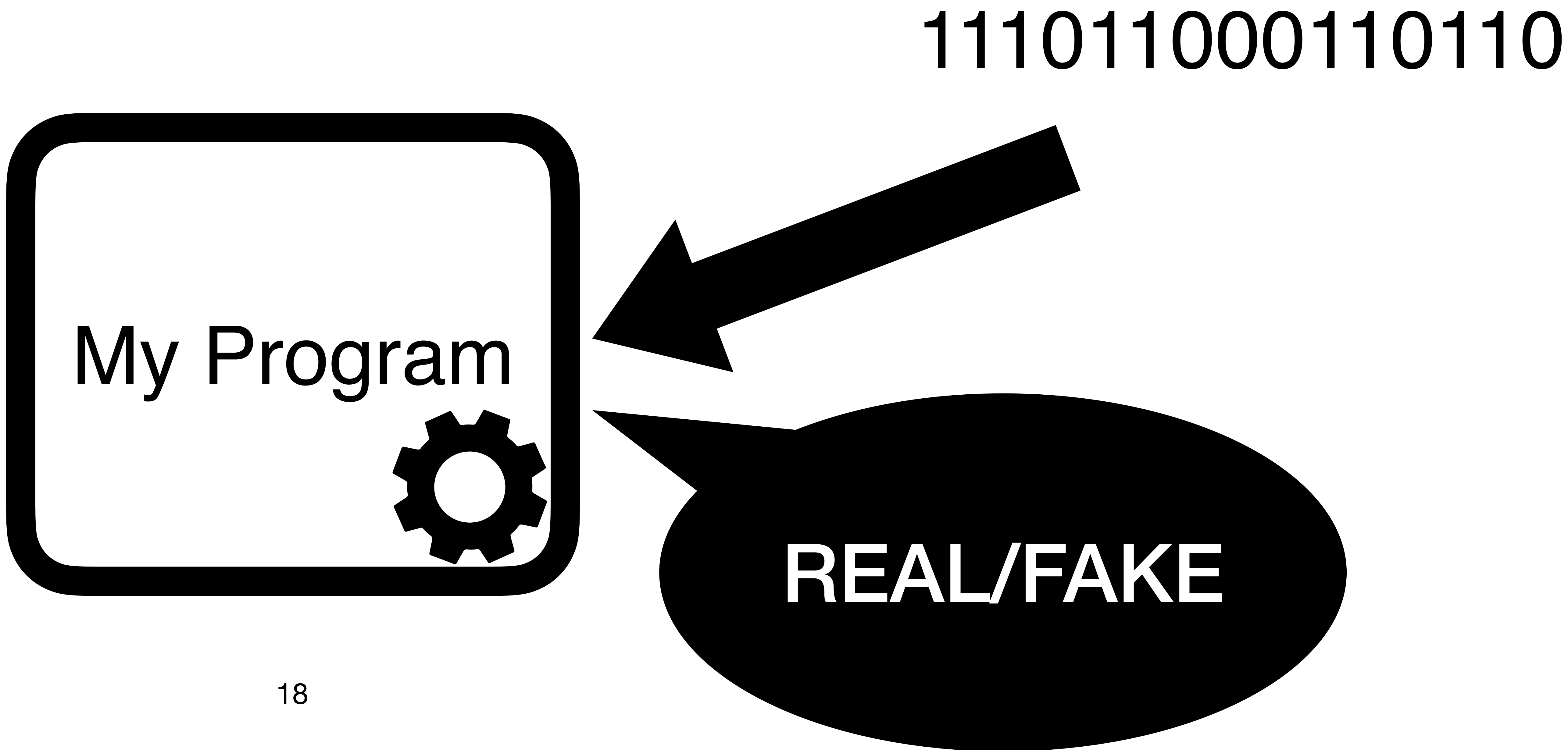
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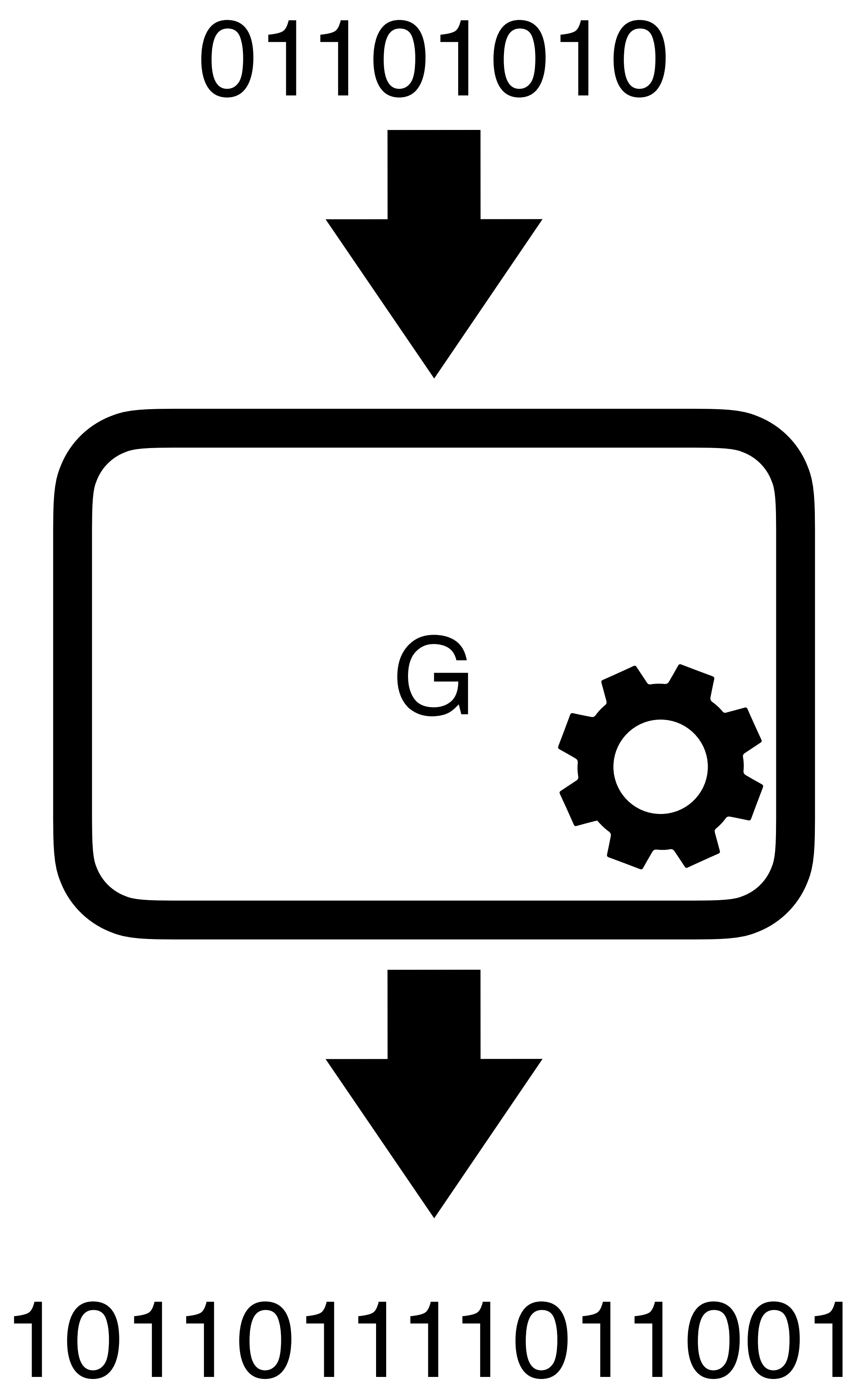
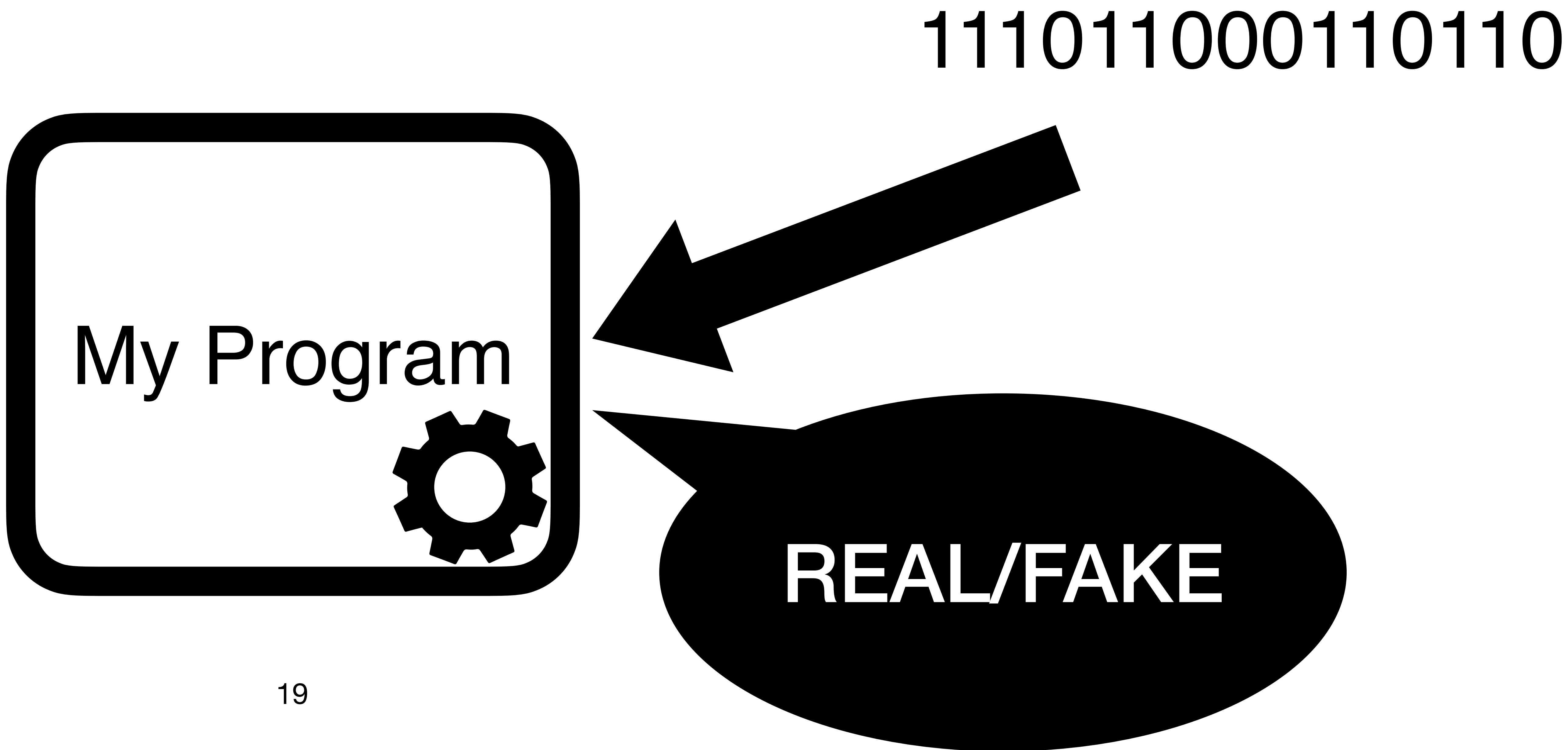
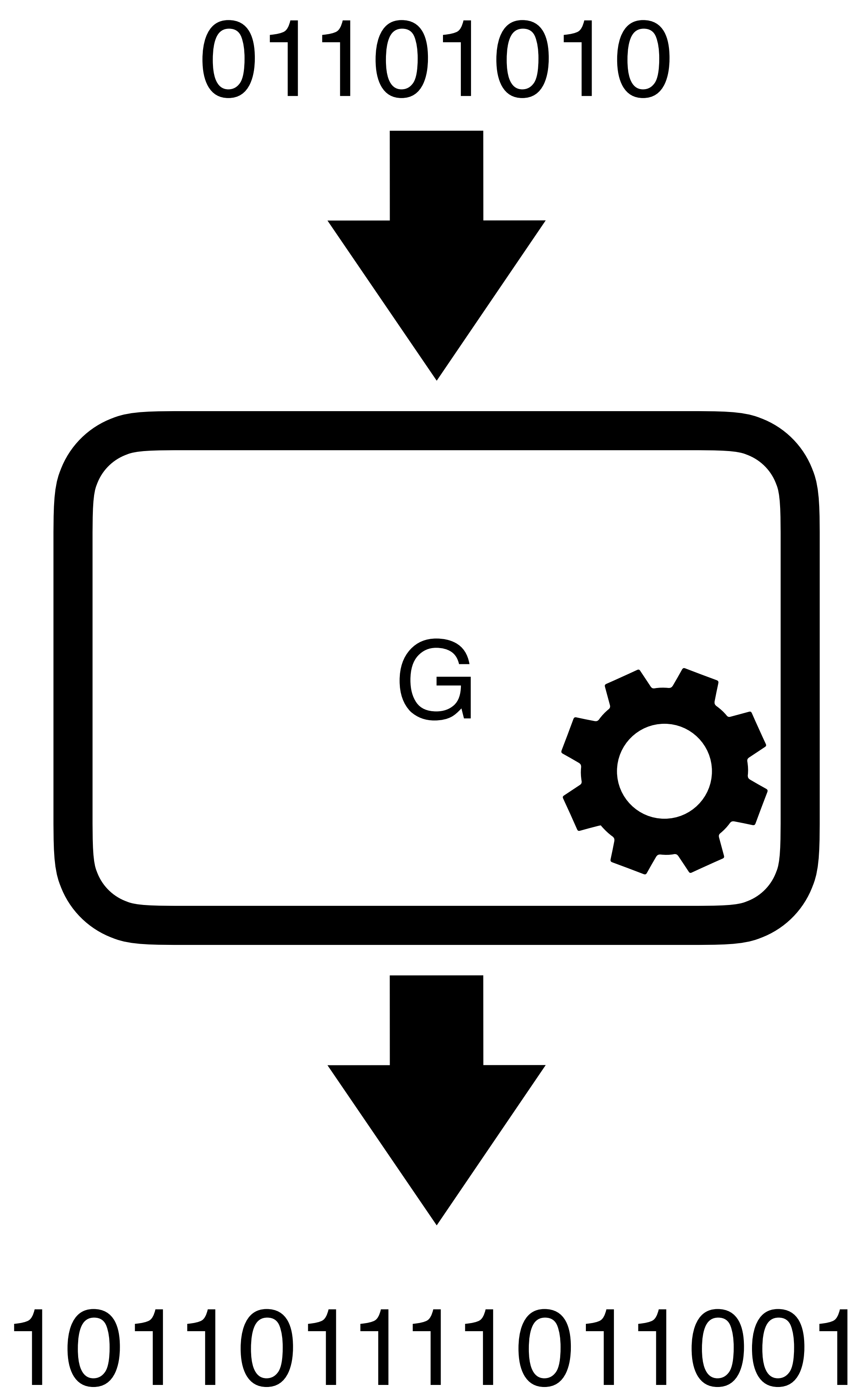


G is a PRG if *no* program can reliably win this game





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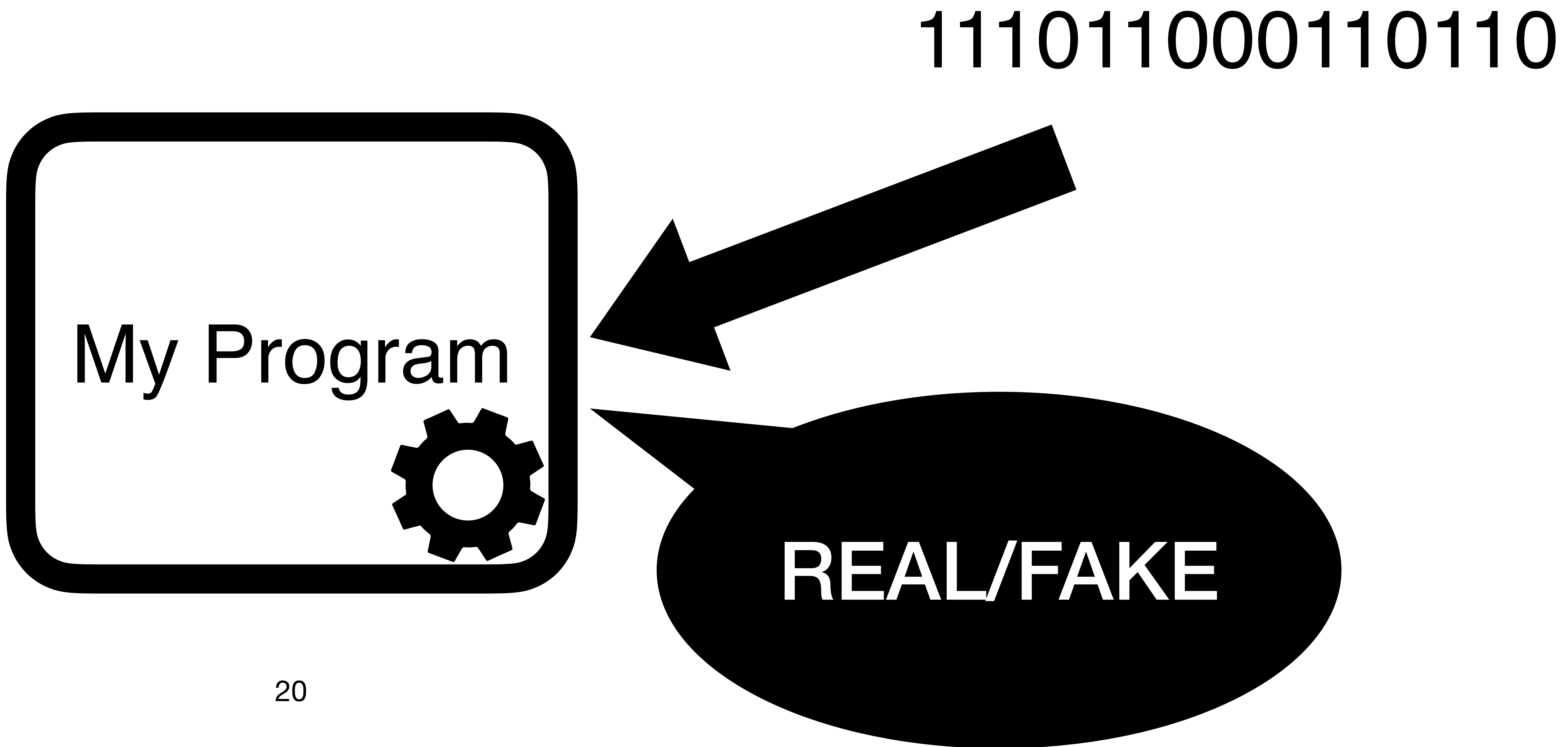
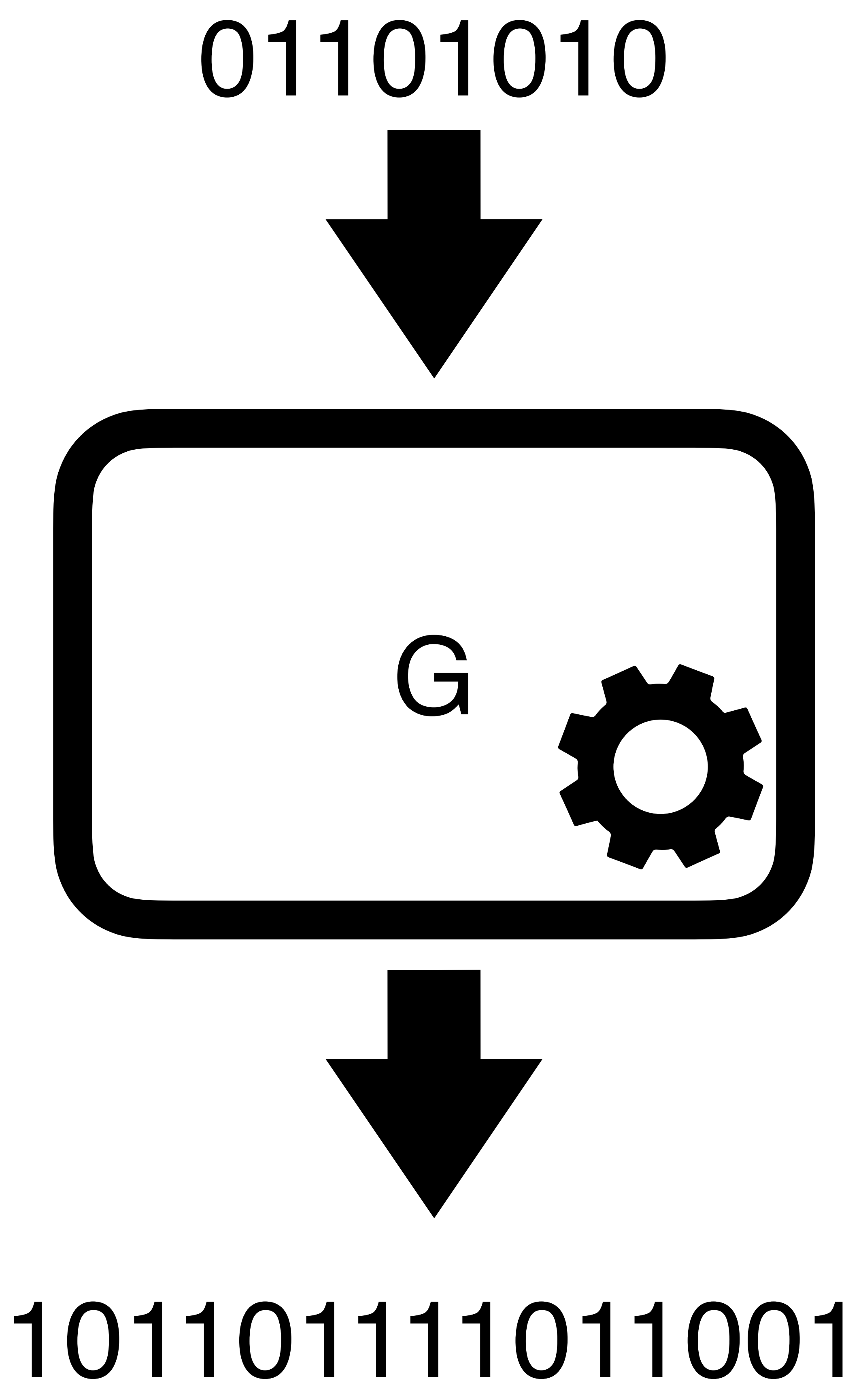


We believe that PRGs exist



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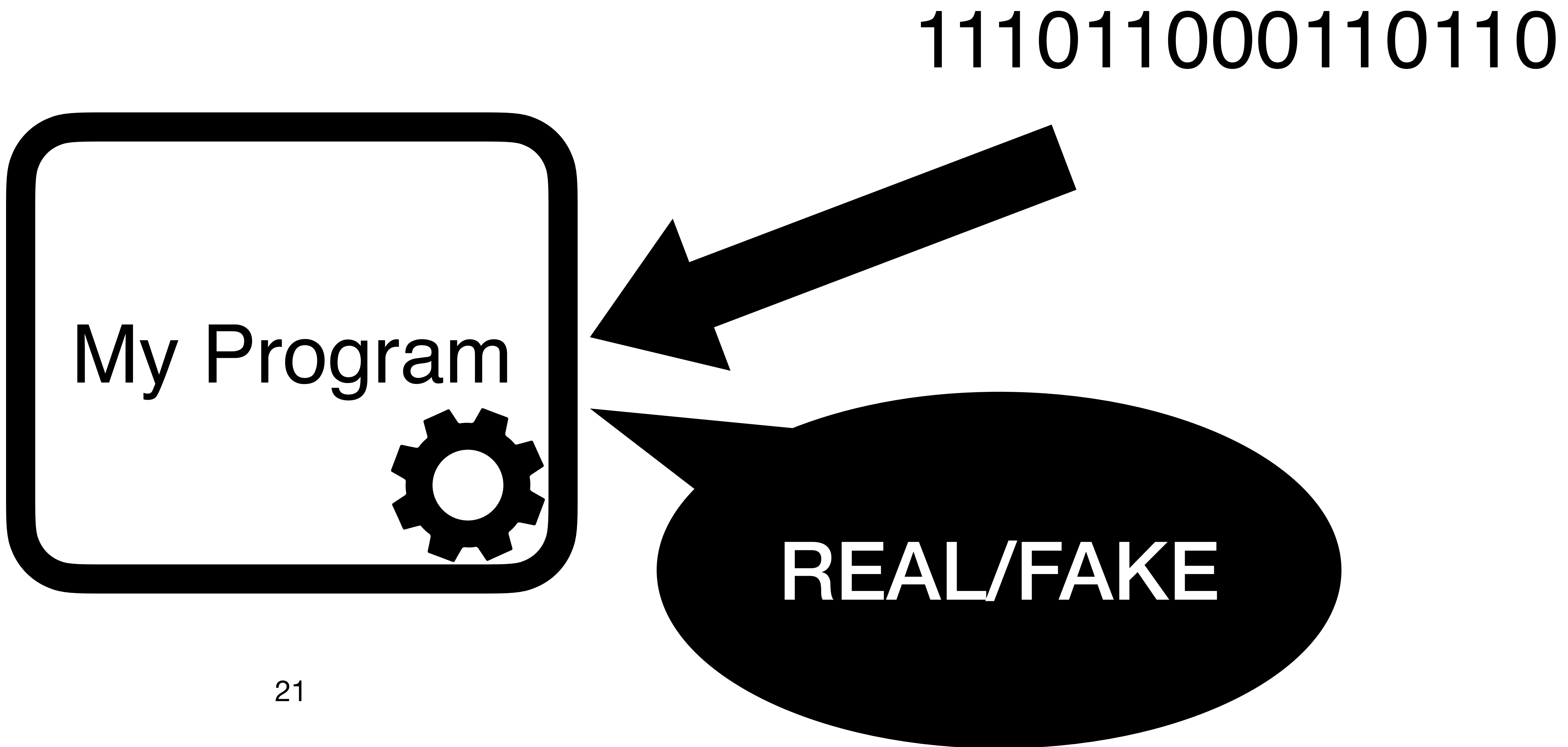

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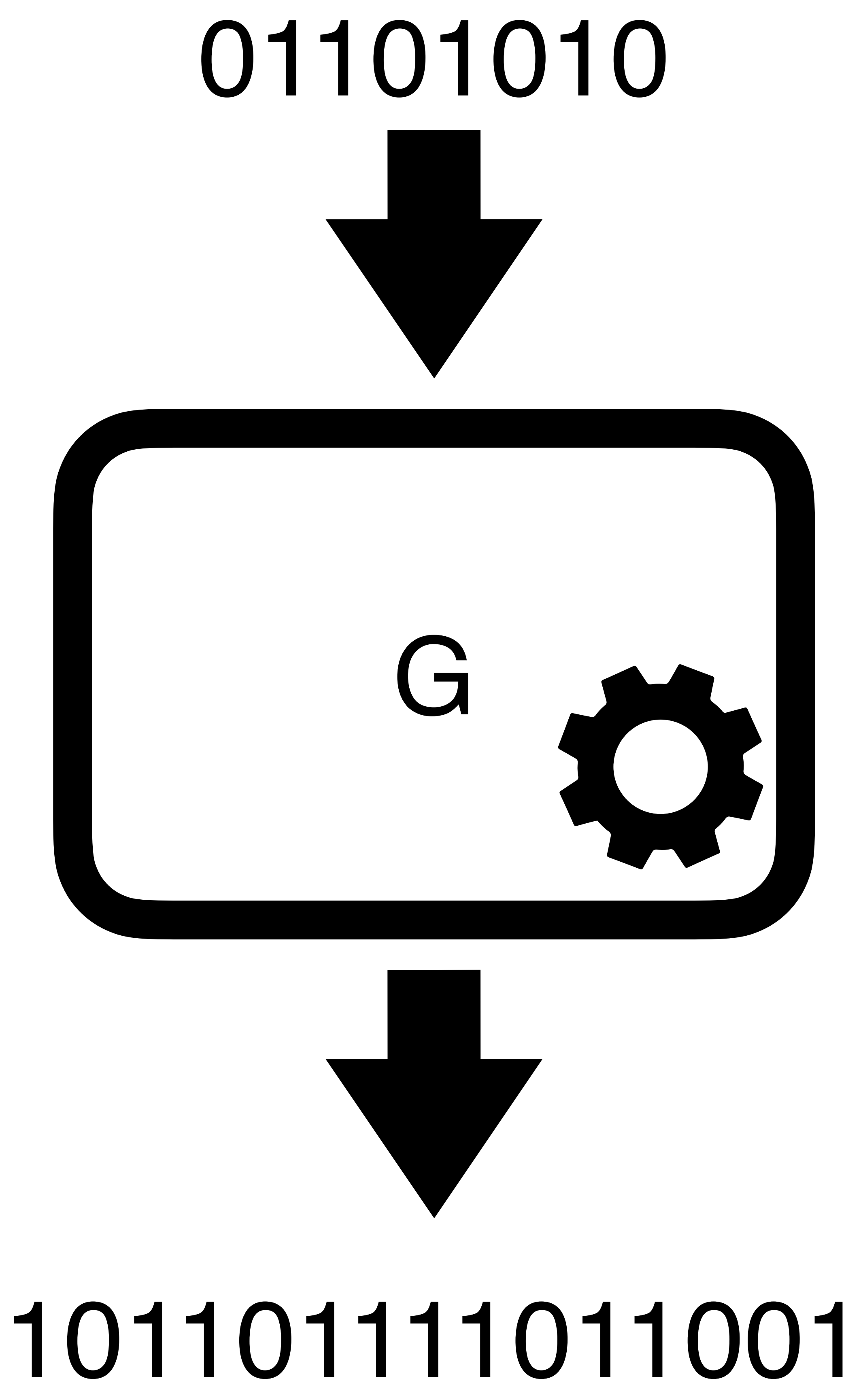

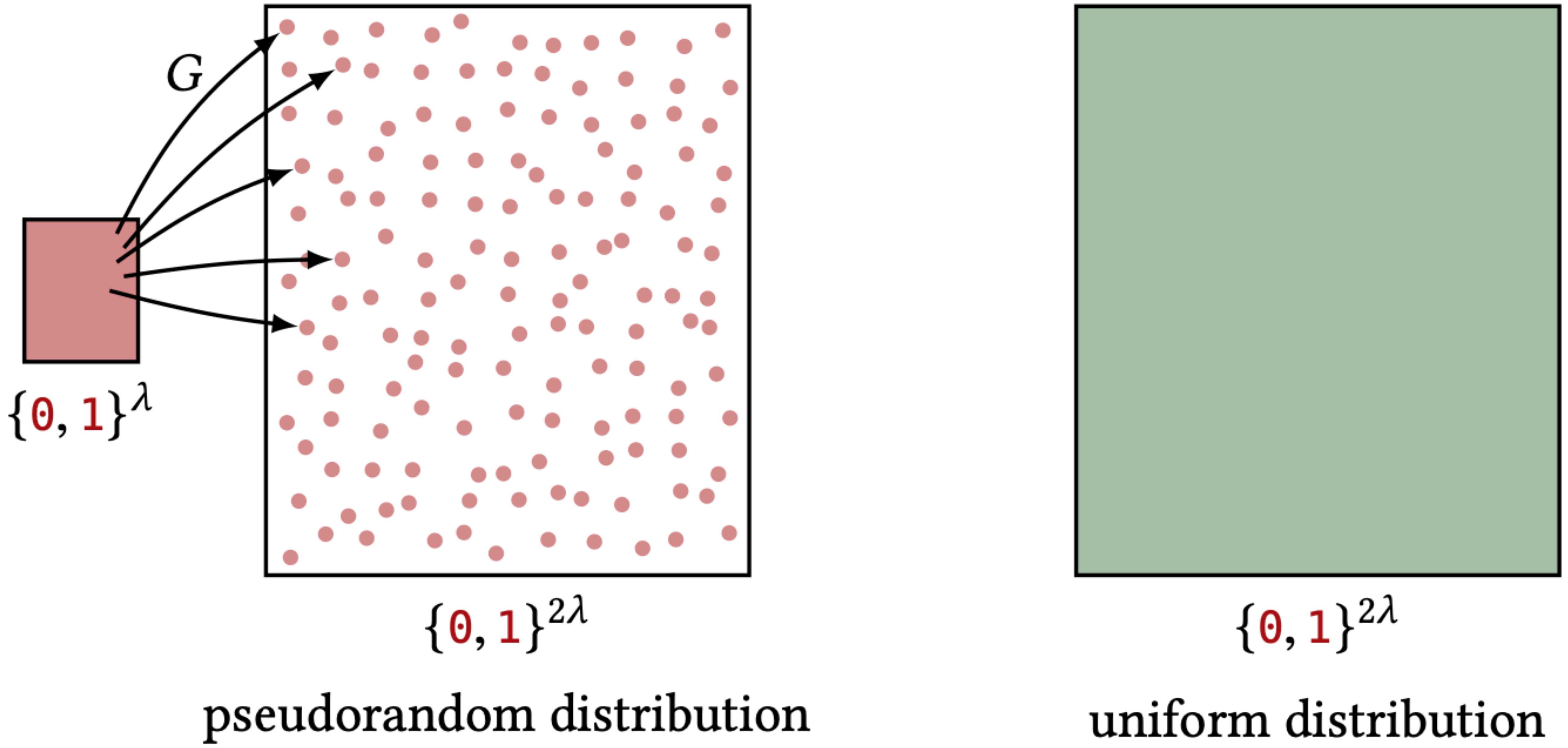

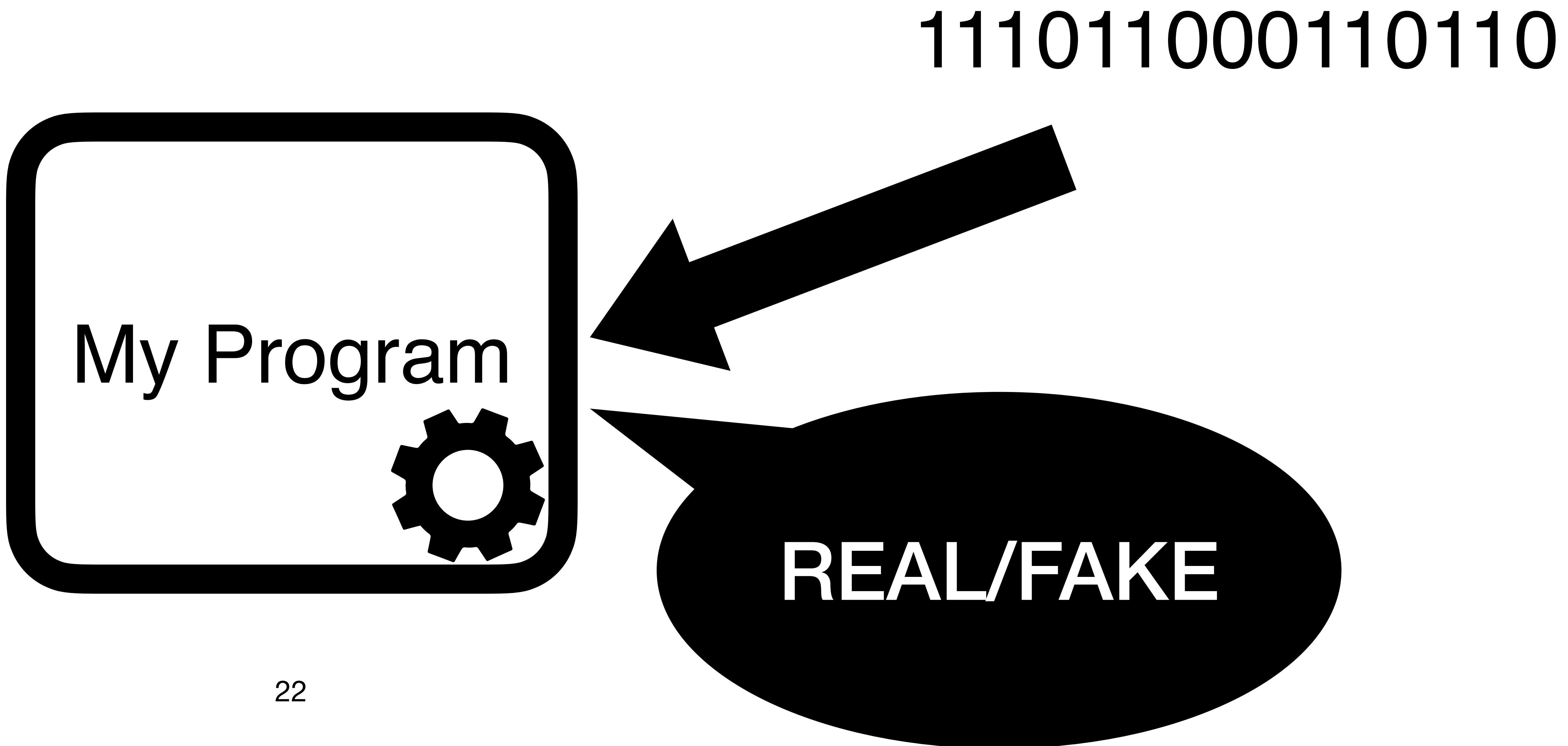

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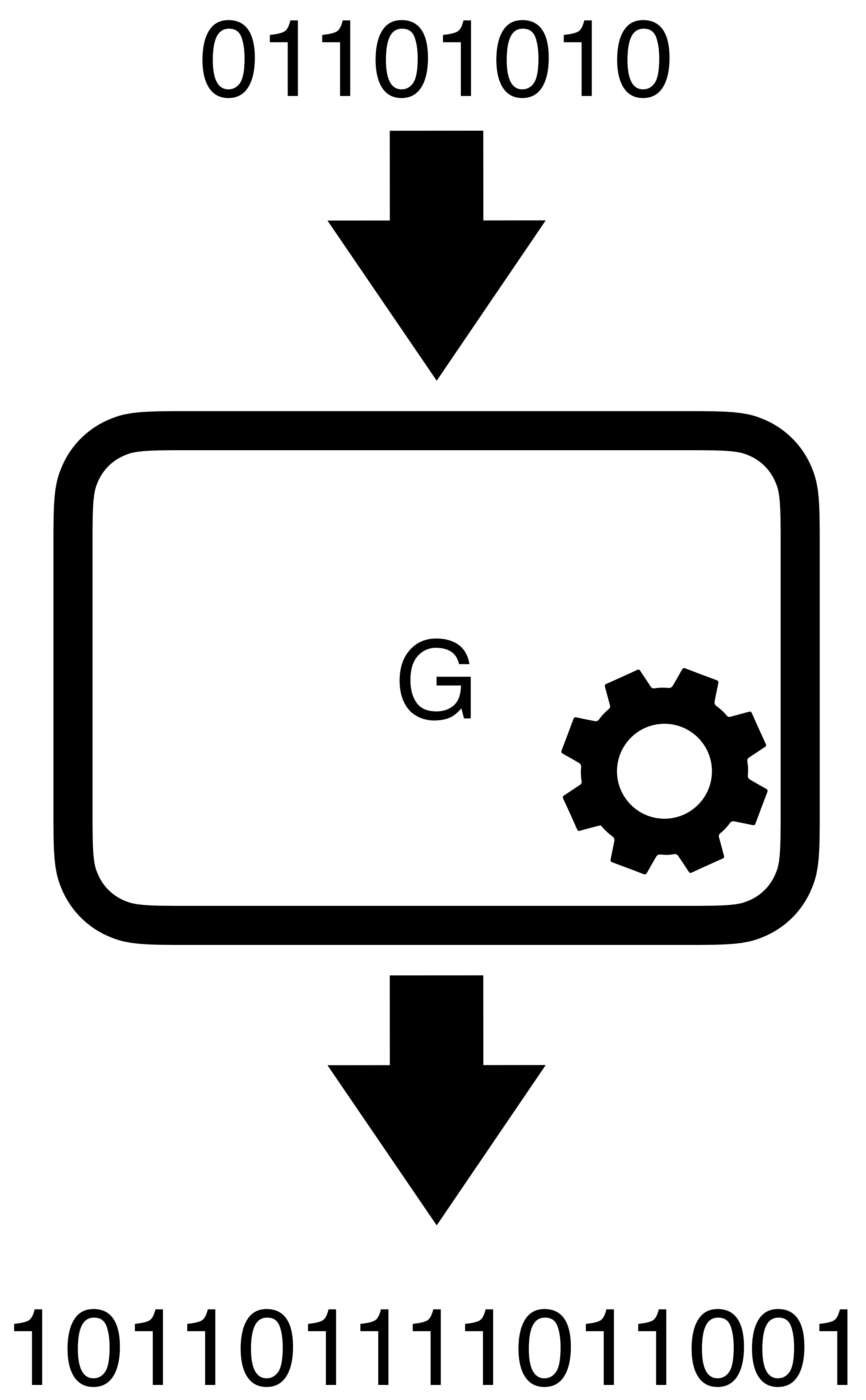
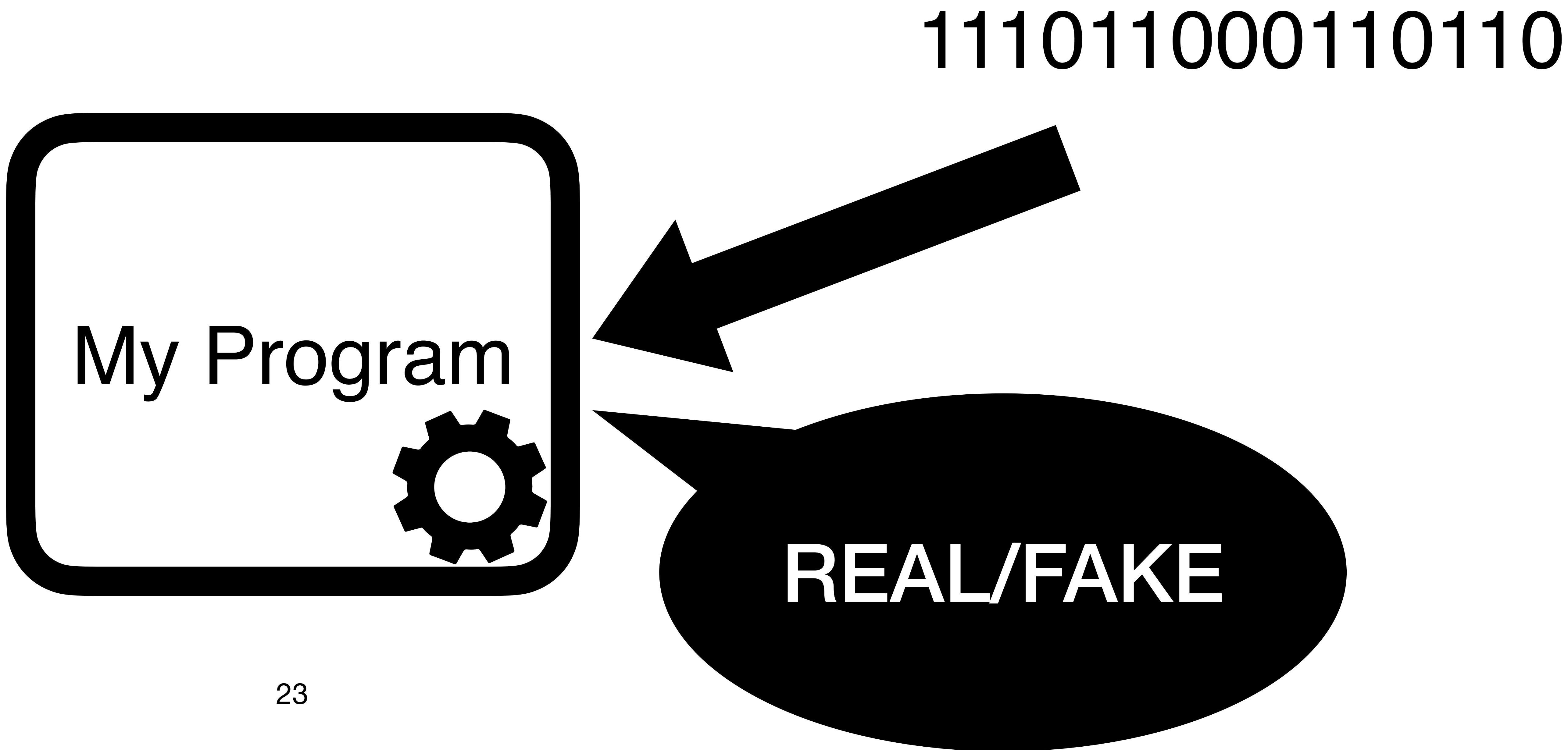


Goal: Make this more precise



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Negligible Function

A function μ is **negligible** *if for any positive polynomial p there exists an N such that for all $n > N$:*

$$\mu(n) < \frac{1}{p(n)}$$

"μ approaches zero really fast"

Probability Ensemble

A probability ensemble is a family of probability distributions indexed by the natural numbers.

We typically call this index the ***security parameter,*** λ

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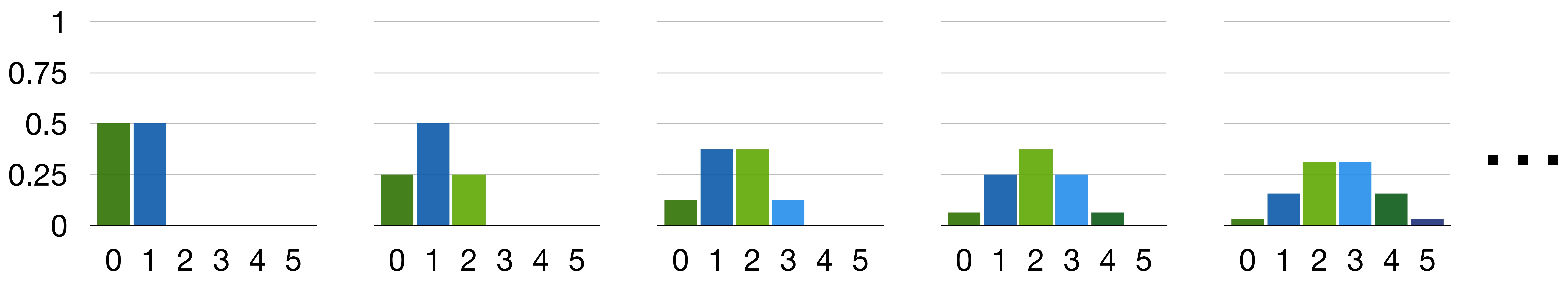

Sum of outcome of λ coin tosses

Indistinguishability

$$\left\{ s \stackrel{?}{=} 0^\lambda \;\middle|\; s \leftarrow_{\$} \{0,1\}^\lambda \right\}_\lambda \qquad \{\, \mathit{false} \,\}_\lambda$$

These ensembles are “the same”

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Imagine showing samples of one ensemble to an adversary. Could they guess which was sampled?

Indistinguishability

Let X, Y be two probability ensembles, and let A be an arbitrary (probabilistic) program that outputs 0 or 1. A's **advantage** is as follows:

$$\text{Advantage}_A(\lambda) = \left| \Pr\left[b = 1 \;\middle|\; \begin{array}{l} x \leftarrow_\$ X_\lambda \\ b \leftarrow A(1^\lambda, x) \end{array} \right] - \Pr\left[b = 1 \;\middle|\; \begin{array}{l} y \leftarrow_\$ Y_\lambda \\ b \leftarrow A(1^\lambda, y) \end{array} \right] \right|$$

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We say that X, Y are **indistinguishable,** written $X \approx Y$ if for every polynomial-time program A:

$\text{Advantage}_A(\lambda)$ is negligible

best strategy is only negligibly better than guessing

Indistinguishability

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These ensembles are "the same"

They are indistinguishable†

† In fact, they are **statistically close**, which is even stronger

PRG security

Let G be a poly-time deterministic algorithm that on an input of length λ outputs a string of length $\lambda + s(\lambda)$.

G is a PRG if $s(\lambda)$ is always positive, and:

$$\left\{ G(k) \;\middle|\; k \leftarrow_\$ \{0,1\}^{\lambda} \right\}_{\lambda} \approx \left\{ r \;\middle|\; r \leftarrow_\$ \{0,1\}^{\lambda+s(\lambda)} \right\}_{\lambda}$$

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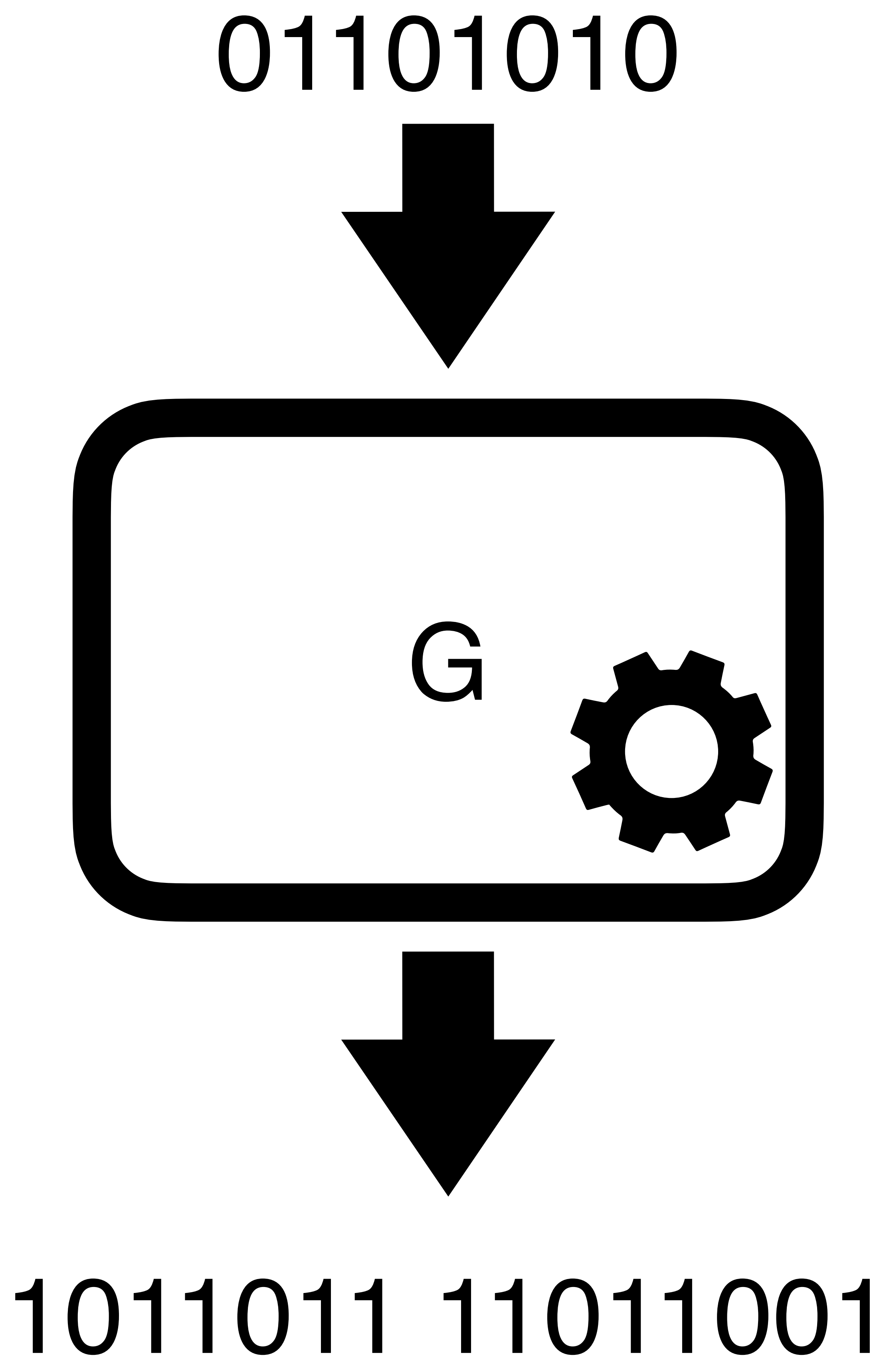
"If seed k is uniform and hidden, then $G(k)$ looks uniform"

Stretching the output of a PRG



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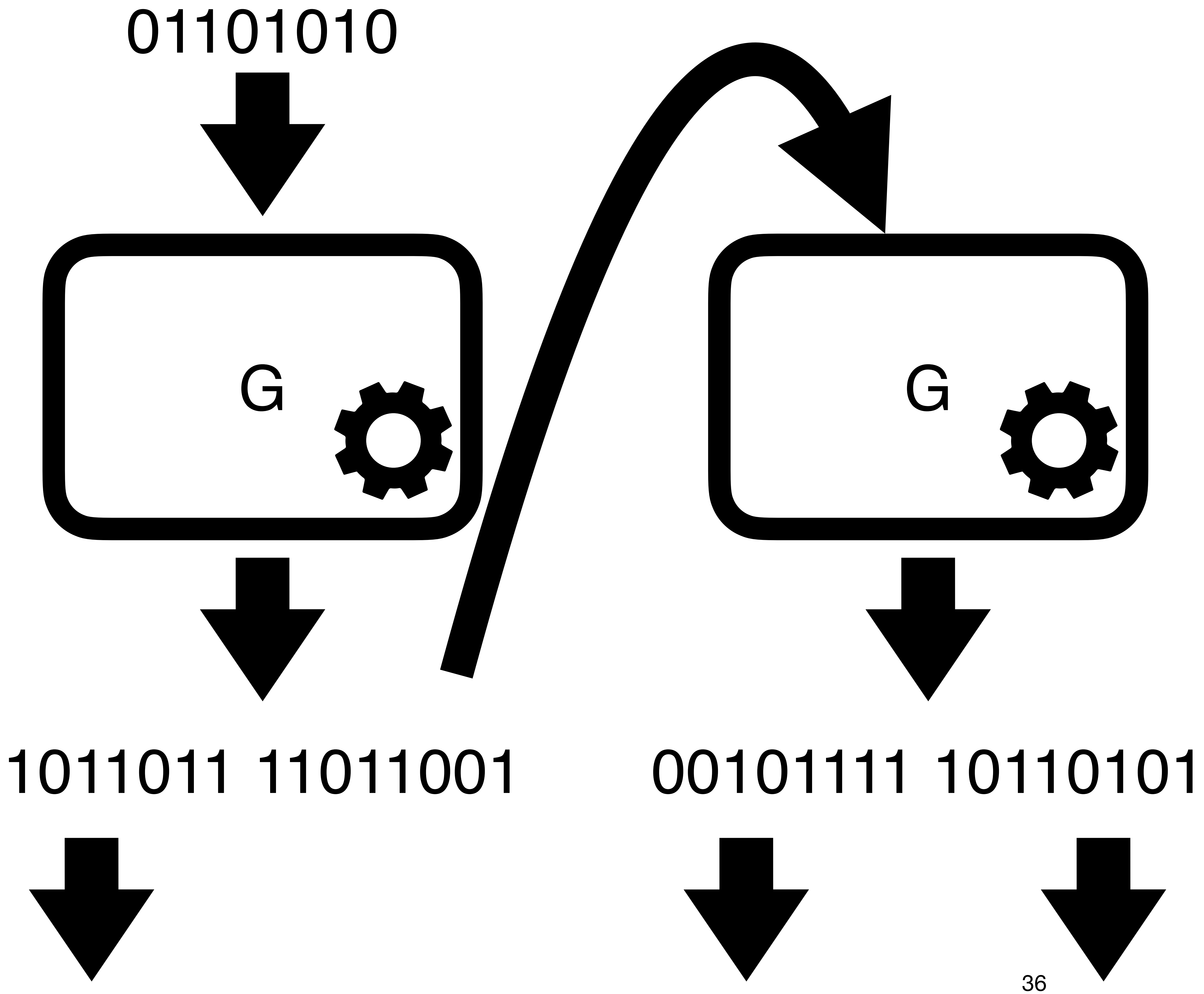

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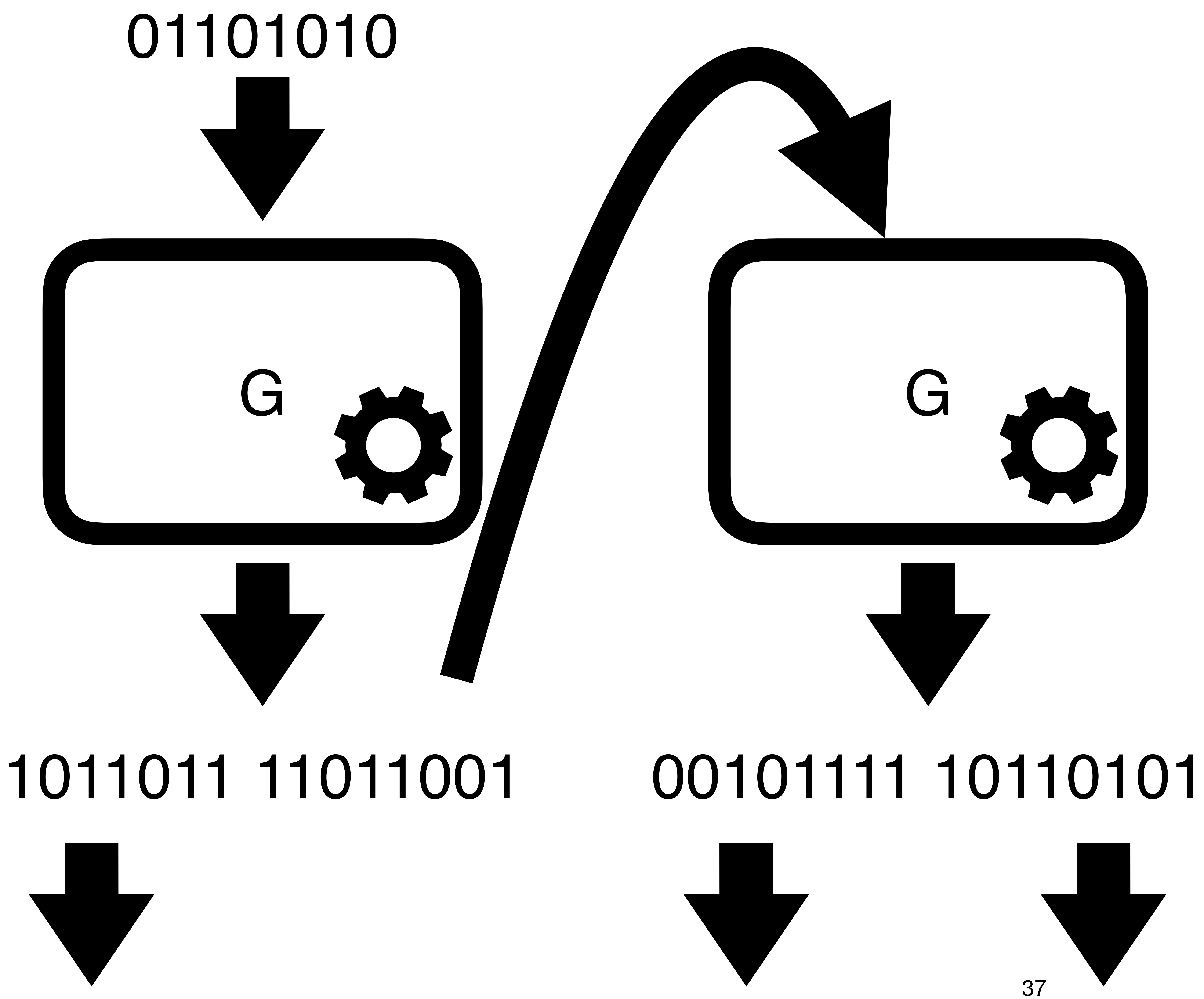
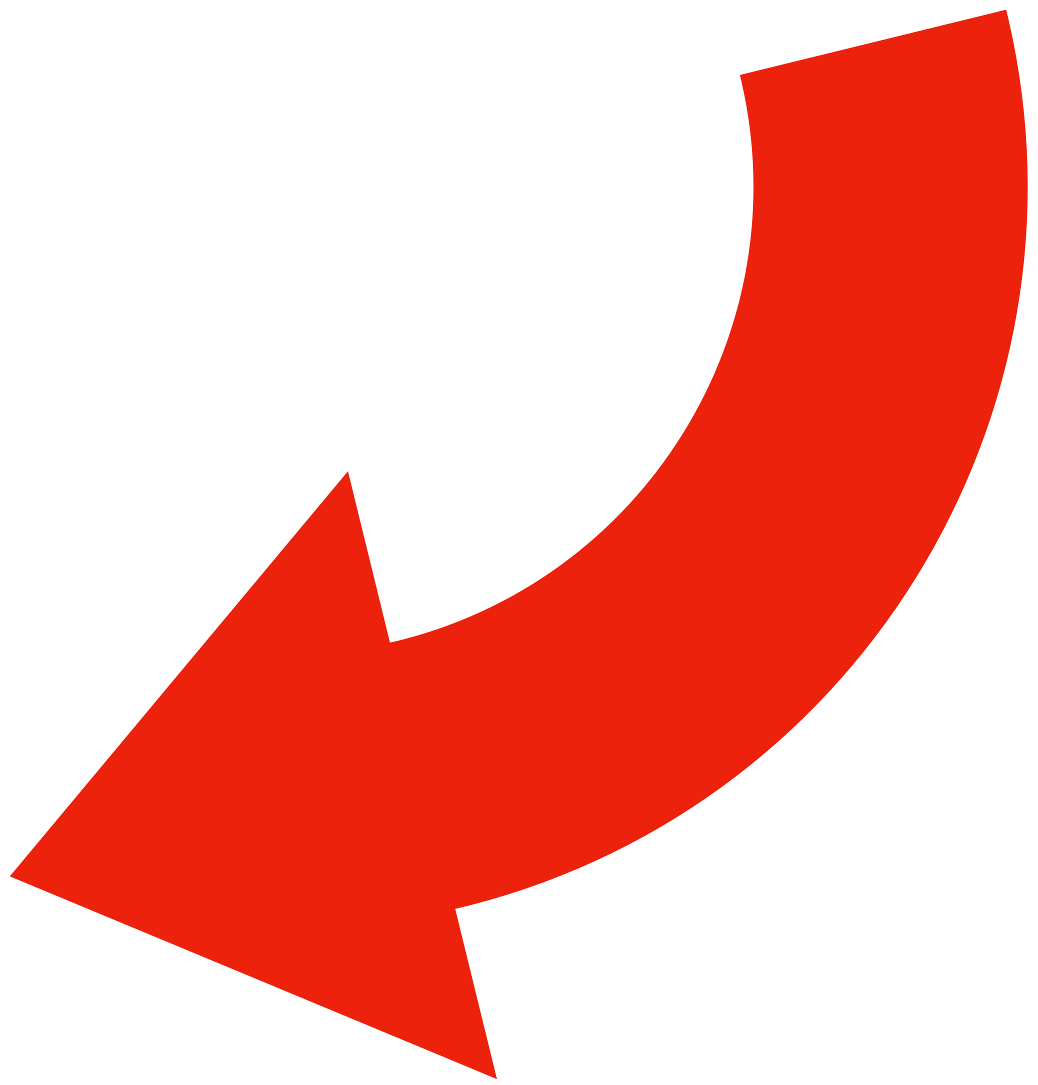
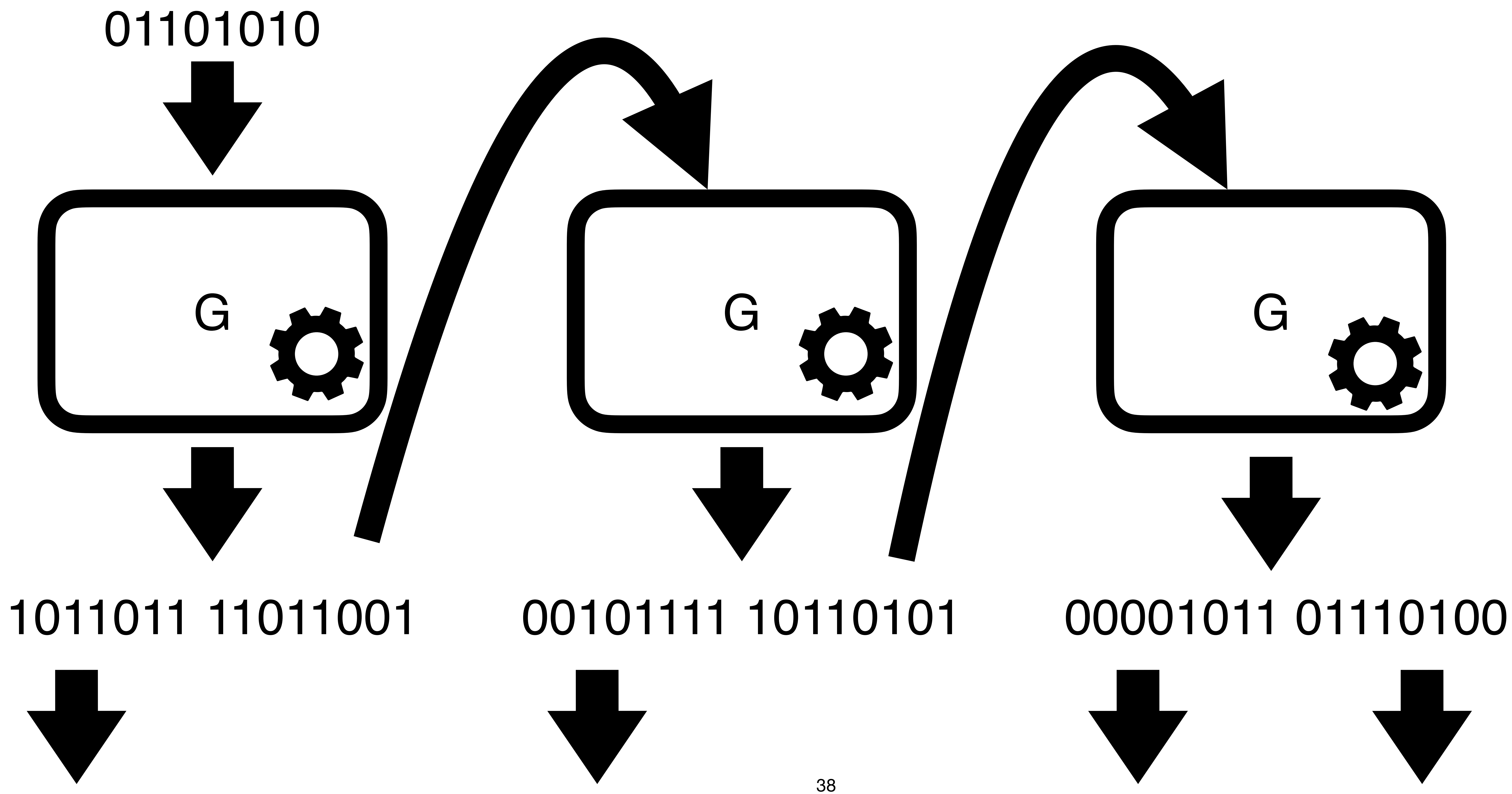
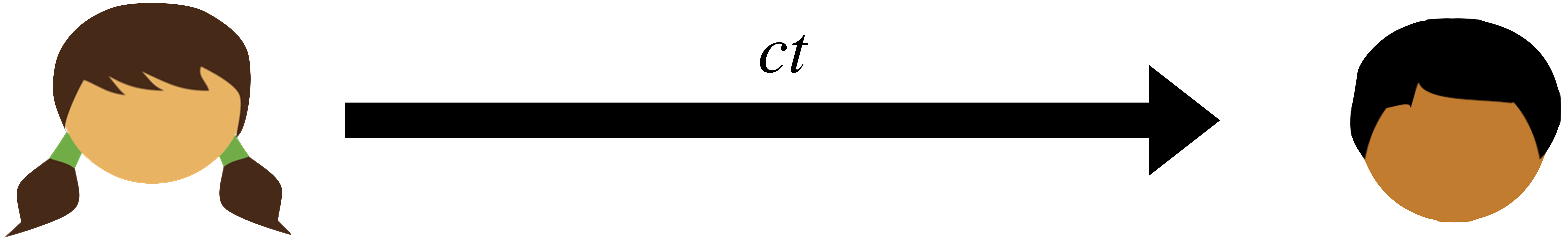


This is a secure PRG



Repeatable any polynomial number of times



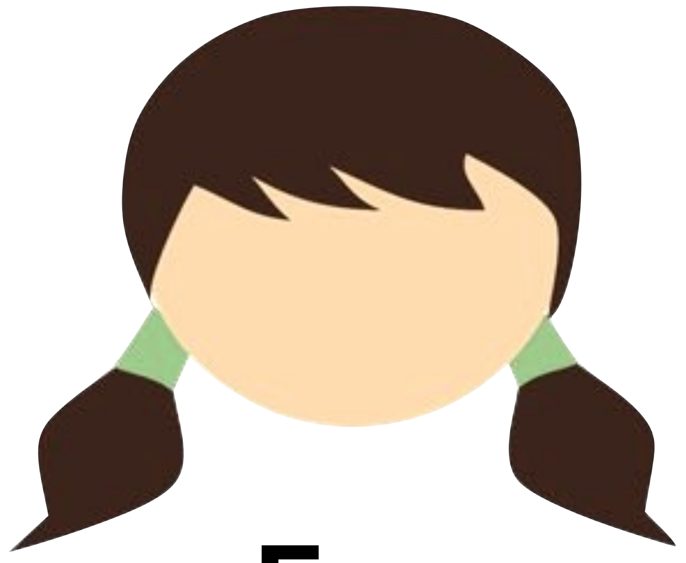


Alice

$m \in \{0,1\}$

$k \leftarrow_{\$} \{0,1\}$

$ct \leftarrow m \oplus k$



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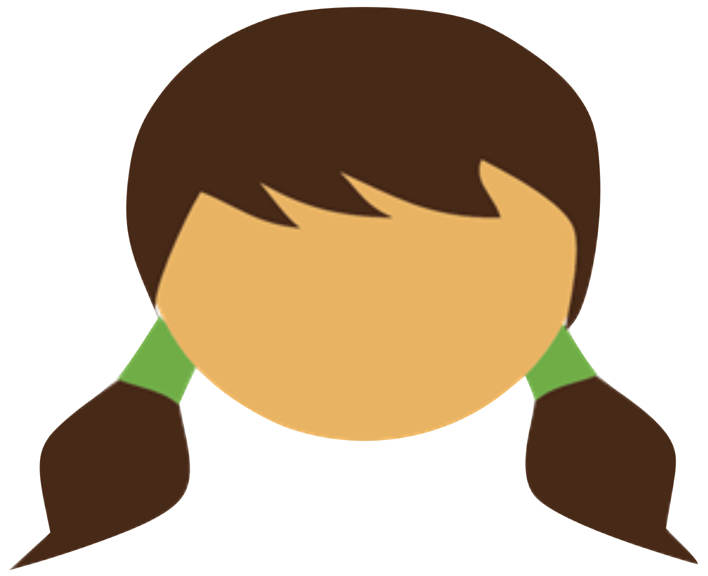
Question: *what if Alice wants to send more than one bit?*



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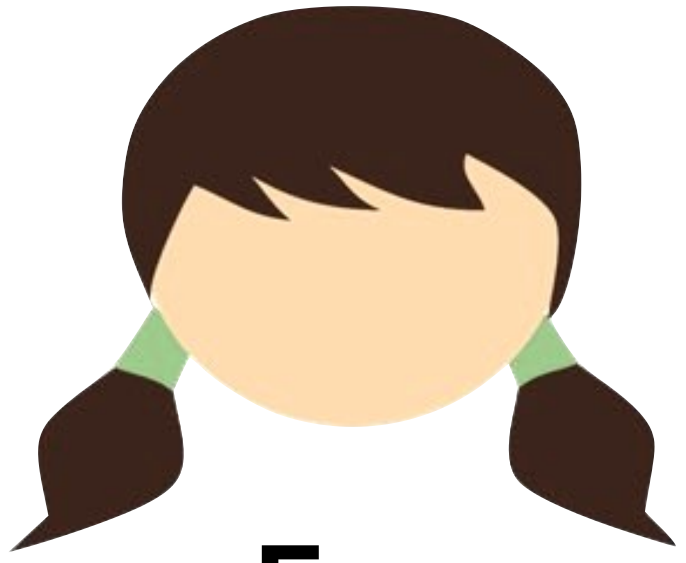

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Question: *what if Alice wants to send more than one bit?*

Answer: *Alice and Bob can exchange a short PRG seed, then expand it (effectively) indefinitely*

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